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Some Results on R-KKM Mappings and R-KKM Selections and Their Applications

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First some intersection theorems based on R-KKM mappings are obtained and then their application to minimax inequalities is given. © 1999 Academic Press

1. INTRODUCTION

The KKM theorem [5] and its generalizations turned out to be a powerful tool dealing with several nonlinear problems. Horvath [2] generalized the KKM theorem replacing the usual convexity assumptions by the topological notion of the contractibility. Motivated by the work of Horvath, Bardaro and Ceppitelli [1] generalized minimax inequalities for functions taking values in ordered vector spaces. At the same time, based on the notion of a convex hull operation introduced by Komiya [6], Joo' [3] and Joo' and Kassay [4] applied the concept of the pseudoconvexity obtaining some intersection theorems with their applications to minimax inequalities. Recently, the author [8–13] obtained some intersection theorems and their applications to minimax inequalities in generalized H-spaces. The aim of this paper is first to obtain an intersection theorem involving R-KKM selections, second to apply it to establish another intersection theorem involving R-KKM mappings, and finally, as a result, to derive some minimax inequalities. The obtained results generalize some results [1, 2, 7–12].

Let X be a topological space without linear structure, $P(X)$ denote the power set of X, and $\langle X \rangle$ the family of all nonempty finite subsets of X. Let Δ^n denote a standard $(n-1)$ simplex $(e_1, \ldots, e_2, \ldots, e_n)$ in R^n.

Joo' [3] introduced, based on the notion of a convex space initiated by Komiya [6], the concept of a pseudoconvex space.

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DEFINITION 1.1. A triple $(X, h, \{p\})$ is called a pseudoconvex space if:

(i) X is a topological space and h: $P(X) \to P(X)$ is a convex hull operation satisfying the properties: $h(\emptyset) = \emptyset$, $h(\{x\}) = \{x\}$ for all $x \in X$, $h(A) = \cup\{h(F)$: $F \in \langle A \rangle\}$ for $A \subset X$, and $h(h(A)) = h(A)$ for $A \subset X$.

(ii) p: $\Delta^n \to h(F)$ is a continuous mapping from Δ^n onto $h(F)$, where Δ^n is a standard simplex of R^n and $F \in \langle X \rangle$.

(iii) For each $F \in \langle X \rangle$, p is convex hull preserving in the sense that $p(e_{i1}, e_{i2}, \dots, e_{ik}) = h(\{x_{i1}, x_{i2}, \dots, x_{ik}\})$ for all subsimplex $(e_{i1}, e_{i2}, \dots, e_{ik}) \subset (e_1, e_2, \dots, e_n)$ and a subset $\{x_{i1}, x_{i2}, \dots, x_{ik}\}$ of $\{x_1, x_2, \dots, x_n\}$ for $\{i1, i2, \dots, ik\} \subset \{1, 2, \dots, n\}$.

A subset $K \subset X$ is called h-convex if $h(K) = K$.

DEFINITION 1.2. A triple $(X, H, \{p\})$ is said to be a generalized H-space ($G{-}H$-space) [8] if X is a topological space and H: $\langle X \rangle \to P(X) \setminus \{\emptyset\}$ is a mapping such that the following assumptions hold:

(i) For each $F, G \in \langle X \rangle$, there exists $F_1 \subset F$ such that $F_1 \subset G$ implies $H(F_1) \subset H(G)$.

(ii) For $F = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$, there exists a continuous mapping p: $\Delta^n \to H(F)$ such that for $\{i1, i2, \dots, ik\} \subset \{1, 2, \dots, n\}$, we have

$$p(e_{i1}, e_{i2}, \dots, e_{ik}) \subset H(\{x_{i1}, x_{i2}, \dots, x_{ik}\}),$$

where $\{x_{i1}, x_{i2}, \dots, x_{ik}\} \subset F$.

A subset K of X is called generalized H-convex ($G{-}H$-convex) if for each $F \in \langle X \rangle$ there exists a subset F_1 of F such that $F_1 \subset K$ implies $H(F_1) \subset K$.

DEFINITION 1.3. Let $(X, H, \{p\})$ be a $G{-}H$-space and T: $X \to P(X)$ a multivalued mapping. T is said to be a relatively KKM (R-KKM) mapping if for any $F = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ there exists a subset $F_1 = \{x_{i1}, x_{i2}, \dots, x_{ik}\}$ of F such that

$$p(e_{i1}, e_{i2}, \dots, e_{ik}) \subset \bigcup_{j=1}^{k} Tx_{ij}$$

for a subsimplex $(e_{i1}, e_{i2}, \dots, e_{ik})$ of $(e_1, e_2, \dots, e_n) = \Delta^n$ for $\{i1, i2, \dots, ik\} \subset \{1, 2, \dots, n\}$.

DEFINITION 1.4 ([8]). Let $(X, H, \{p\})$ be a $G{-}H$-space, and $M_1, M_2, \dots, M_n$ be subsets of X. A subset $F = \{x_1, x_2, \dots, x_n\}$ of n elements of X is said to be a generalized KKM (G-KKM) selection for $M_1, M_2, \dots, M_n$ if for $\{i1, i2, \dots, ik\} \subset \{1, 2, \dots, n\}$, we have

$$H(\{x_{i1}, x_{i2}, \dots, x_{ik}\}) \subset \bigcup_{j=1}^{k} M_{ij},$$

where elements $x_1, x_2, \dots, x_n$ are not necessarily distinct.

This generalizes the notion of a KKM selection introduced by Joo' and Kassay [4].

DEFINITION 1.5 [8]. Let $(X, H, \{p\})$ be a G−H-space, $x_1, x_2, \dots, x_n$ be n elements of X and, $M_1, M_2, \dots, M_n$ subsets of X. Elements $x_1, x_2, \dots, x_n$ are called a relative KKM (R-KKM) selection for $M_1, M_2, \dots, M_n$ if for $\{i1, i2, \dots, ik\} \subset \{1, 2, \dots, n\}$, we have

$$p(e_{i1}, e_{i2}, \dots, e_{ik}) \subset \bigcup_{j=1}^{k} M_{ij},$$

where $x_1, x_2, \dots, x_n$ are not necessarily distinct, and $(e_{i1}, e_{i2}, \dots, e_{ik})$ is a standard $n - 1$ subsimplex of $(e_1, e_2, \dots, e_n)$ in R^n.

Clearly, every KKM selection [4] is an R-KKM selection.

Note that Definition 1.5 implies that

$$p(e_{i1}, e_{i2}, \dots, e_{ik}) \subset \bigcup_{j=1}^{k} \left(M_{ij} \cap H(\{x_{i1}, x_{i2}, \dots, x_{ik}\}\right),$$

where $\{x_{i1}, x_{i2}, \dots, x_{ik}\} \subset \{x_1, x_2, \dots, x_n\}$.

Let X and Y be two sets, and S: $X \to P(Y)$ be a multivalued mapping. Then S^{-1}: $Y \to P(X)$ and S^*: $Y \to P(X)$ are defined by $x \in S^{-1}(y)$ iff $y \in Sx$ and $S^*y = X \setminus S^{-1}y$, respectively. Clearly, $x \in S^*y$ iff $y \notin Sx$.

2. R-KKM MAPPING THEOREMS

This section deals with the intersection theorems involving R-KKM mappings. The first theorem generalizes mildly [2, Theorem 1] to the case of R-KKM selections in a G−H-space.

THEOREM 2.1. *Let $(X, H, \{p\})$ be a G−H-space, $F = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$, and $M_1, M_2, \dots, M_n$ closed subsets of X. Suppose that $\{x_1, x_2, \dots, x_n\}$ is a* R-KKM *selection for $M_1, M_2, \dots, M_n$ relative to a simplex $(e_1, e_2, \dots, e_n) = \Delta^n$. Then we have $\bigcap_{i=1}^{n} M_i \neq \emptyset$.*

Proof. Since $(X, H, \{p\})$ is a G−H-space and $F = \{x_1, x_2, \dots, x_n\}$ is a R-KKM selection for the subsets $M_1, M_2, \dots, M$, we have

$$p(e_{i1}, e_{i2}, \dots, e_{ik}) \subset \bigcup_{j=1}^{k} M_{ij} \quad \text{for } \{i1, i2, \dots, ik\} \subset \{1, 2, \dots, n\},$$

where $(e_{i1}, e_{i2}, \dots, e_{ik})$ is a subsimplex of $(e_1, e_2, \dots, e_n) = \Delta^n$ of R^n, and $\{x_{i1}, x_{i2}, \dots, x_{ik}\} = F_1 \subset F$. If we set $E_i = p^{-1}(M_i)$ for $i = 1, 2, \dots, n$, then E_i is closed in Δ^n by the closedness of M_i and the continuity of p. Before we can apply the KKM theorem [5], we need to show that $\mathrm{co}(e_{i1}, e_{i2}, \ldots, e_{ik}) \subset \bigcup_{j=1}^{k} E_{ij}$ for $\{i1, i2, \ldots, ik\} \subset \{1, 2, \ldots, n\}$. From the above arguments, it follows that

$$\mathrm{co}(e_{i1}, e_{i2}, \ldots, e_{ik}) \subset p^{-1}\left(\bigcup_{j=1}^{k} M_{ij}\right) = \bigcup_{j=1}^{k} \left(p^{-1}(M_{ij})\right) = \bigcup_{j=1}^{k} E_{ij}.$$

Hence, $\bigcap_{i=1}^{n} E_i \neq \emptyset$. If an element $t_0 \in \bigcap_{i=1}^{n} E_i$, then $t_0 \in E_i = p^{-1}(M_i)$ for all $i = 1, 2, \ldots, n$. As a result, $p(t_0) \in M_i$ for all $i = 1, 2, \ldots, n$, that means, $p(t_0) \in \bigcap_{i=1}^{n} M_i$. This completes the proof.

THEOREM 2.2. *Let X be a topological space, $(Y, H, \{p\})$ a compact G–H-space, and T: $X \to P(Y) \setminus \{\emptyset\}$ a closed valued mapping. Suppose that T is a* R-KKM *mapping. Then we have $\bigcap_{x \in X} Tx \neq \emptyset$.*

Proof. The proof is immediate from an application of Theorem 2.1.

THEOREM 2.3. *Let $(X, H, \{p\})$ be a compact G–H-space and S, T: $X \to P(X)$ two multivalued mappings such that*:

(i) *Tx is closed and $Sx \subset Tx$ for all $x \in X$.*

(ii) *$x \in Sx$ for all $x \in X$.*

(iii) *S^*x is G–H-convex for all $x \in X$.*

Then $\bigcap_{x \in X} Tx \neq \emptyset$.

Proof. The proof follows from an application of Theorem 2.1 if we show that T is a R-KKM mapping, that is,

$$p(e_{i1}, e_{i2}, \ldots, e_{ik}) \subset \bigcup_{j=1}^{k} Tx_{ij} \quad \text{for } \{i1, i2, \ldots, ik\} \subset \{1, 2, \ldots, n\},$$

where $(e_{i1}, e_{i2}, \ldots, e_{ik})$ is a subsimplex of $(e_1, e_2, \ldots, e_n) = \Delta^n$ and $F_1 = \{x_{i1}, x_{i2}, \ldots, x_{ik}\}$ is a subset of $F = \{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$. Assume that T is not R-KKM. Then, since $(X, H, \{p\})$ is a G–H-space, we have

$$H(\{x_{i1}, x_{i2}, \ldots, x_{ik}\}) \not\subset \bigcup_{j=1}^{k} Tx_{ij}.$$

If an element $y \in H(\{x_{i1}, x_{i2}, \ldots, x_{ik}\}$, then $y \notin Tx_{ij}$ for $x_{ij} \in F_1$ for $j = 1, 2, \ldots, k$. That means, $x_{ij} \in T^*y$, and as a result, $F_1 \subset T^*y$. Given that $Sx \subset Tx$ for all $x \in X$, we have $T^*y \subset S^*y$. This implies $F_1 \subset S^*y$, and by (iii), $H(F_1) \subset S^*y$. Thus, $y \in S^*y$, which implies $y \notin Sy$, a contradiction to (ii). This completes the proof.

THEOREM 2.4. *Let* $(X, H, \{p\})$ *be a compact* $G-H$*-space, and* S, T: $X \to P(X) \setminus \{\emptyset\}$ *multivalued mappings such that*:

(i) $Sx \subset Tx$ *for all* $x \in X$.

(ii) Tx *is* $G-H$*-convex for all* $x \in X$.

(iii) $S^{-1}y$ *is open in* X *for all* $y \in X$.

Then there exists an element $x_0 \in X$ *such that* $x_0 \in Tx_0$. *This is a variant form of Theorem* 2.3.

3. MINIMAX INEQUALITIES

In this section, we apply the results of Section 2 to derive some minimax inequalities, which generalize some minimax results [2] in pseudoconvex spaces and H-spaces.

THEOREM 3.1. *Let* $(X, H, \{p\})$ *be a compact* $G-H$*-space and* f, g: $X \times X \to R$ *two functions such that the following assumptions hold*:

(i) $g(x, y) \leq f(x, y)$ *for all* $(x, y) \in X \times X$.

(ii) $g(x, y)$ *is lower semicontinuous in its second variable, that is, the set* $\{y \in X\colon g(x, y) \leq c \text{ for } c \in R\}$ *is closed*.

(iii) $f(x, y)$ *is* $G-H$*-quasiconcave in its first variable, that is, the set* $\{x \in X\colon f(x, y) > c\}$ *is* $G-H$*-convex*.

Then one of the following alternatives holds:

(A) *There exists an element* y_0 *of* X *such that* $g(x, y_0) \leq c$ *for all* $x \in X$.

(B) *There is an element* $x_0 \in X$ *such that* $f(x_0, x_0) > c$.

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1, *we have*

$$\inf_{y \in X} \sup_{x \in X} g(x, y) \leq \sup_{x \in X} f(x, x).$$

Proof of Theorem 3.1. Let us define two multivalued mappings S and T by $Sx = \{y \in X\colon f(x, y) \leq c \text{ for } c \in R\}$ and $Tx = \{y \in X\colon g(x, y) \leq c \text{ for } c \in R\}$, respectively. Then it follows from (i) that $Sx \subset Tx$. Consider an element $x_0 \in X$ such that $x_0 \notin Sx_0$. Then $f(x_0, x_0) > c$, which proves (B). Next, we consider the case when $x \in Sx$ for all $x \in X$. For $A \in \langle X \rangle$, $A_1 \subset A$, and $S^*y = \{x \in X\colon f(x, y) > c\}$, let $A_1 \subset S^*y$. Then, by (iii), it follows that $H(A_1) \subset S^*y$. Now, by Theorem 2.3, we have $\bigcap_{x \in X} Tx \neq \emptyset$, which implies (A).

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